Math 623 Exam 2 Solutions

1. Set $A = \begin{pmatrix} 3 & 1 & -1 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. Find J_A, the Jordan canonical form for A.

Because A is triangular, we see that $\lambda = 5$ has algebraic multiplicity 1, and $\lambda = 3$ has algebraic multiplicity 2. We calculate $(A - 3I)^2 = \begin{pmatrix} 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $(A - 3I)^3 = \begin{pmatrix} 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $(A - 3I)^4 = \begin{pmatrix} 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Hence $r_0(A, 3) = 4, r_1(A, 3) = 3, r_2(A, 3) = 2, r_3(A_3) = 1, r_4(A_3) = 1$. Hence the Weyr characteristic is $(1, 1, 1)$ and the Segre characteristic is (3). Hence $J_A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

2. Let $A \in M_n(\mathbb{C})$, and let $\lambda \in \mathbb{C}$ be fixed. Prove that the following are equivalent:
(1) λ is NOT an eigenvalue of A;
(2) $w_k(A, \lambda) = 0$ for all $k \geq 1$; and
(3) $w_1(A, \lambda) = 0$.

See Ex. 3.1.1 and 3.1.8. First note that A and the JCF of A they have the same r_k and w_k values by Lemma 3.1.18.
$(1 \to 2)$ Let J be the JCF of A. Then $J - \lambda I$ is an upper triangular matrix with nonzero diagonal entries; hence every power of it is as well, so $r_k(J, \lambda) = n$, for all k, and so $w_k(J, \lambda) = 0$.
$(2 \to 3)$ trivial.
$(3 \to 1)$ Suppose by way of contradiction λ is an eigenvalue. Let J be the JCF of A, ordered so that a λ-block comes first. Then $J - \lambda I$ has only 0 in the first column; hence $r_1(J, \lambda) \leq n - 1$ so $w_1(J, \lambda) \geq 1$.

3. Let $B \in M_n(\mathbb{C})$ be invertible. Suppose that there is some $A \in M_n(\mathbb{C})$ with $B = A^\star A$. Prove there exists a unique $L \in M_n(\mathbb{C})$, such that L is lower triangular, has positive real diagonal entries, and $B = LL^\star$.

See the last exercise in 2.1. Because B is invertible, so is A. We now apply Thm 2.1.14 to write $A = QR$, where A is unitary, R is upper triangular with positive diagonal entries. We have $B = A^\star A = (QR)^\star QR = R^\star Q^\star QR = R^\star R$, since $Q^\star Q = I$. We now set $L = R^\star$ and $L^\star = R$, as desired.

4. Find the rational canonical form and the real Jordan canonical form of $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

We have $P_A(t) = \begin{vmatrix} t-1 & 1 \\ -1 & t-1 \end{vmatrix}^2 = x^4 - 4x^3 + 8x^2 - 8x + 4$. This has complex roots $1 \pm i$, each with multiplicity 2. Hence the real JCF is $\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ and the RCF is $\begin{pmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 4 \end{pmatrix}$.

5. For each $n \geq 4$, find nilpotent matrices $A, B \in M_n$ that are not similar to each other, but have the same minimal polynomial.

See 3.3.P1. Since A, B are nilpotent, their only eigenvalues are 0. Since they have the same minimal polynomial, the largest Jordan block of each must be the same size. Since they are not similar, not all the Jordan blocks must be the same. Hence one solution is $A = J_2(0) \oplus J_2(0) \oplus (J_1(0))^{n-4}$, $B = J_2(0) \oplus (J_1(0))^{n-2}$.

6. Let $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and set $W_J = \begin{bmatrix} \lambda I_n & I_n \\ 0 & \lambda I_n \end{bmatrix}$. Prove that $A \in M_{2n}$ commutes with W_J if and only if $A = \begin{bmatrix} B & C \\ 0 & B \end{bmatrix}$ for some matrices $B, C \in M_n$.

See Ex. 3.4.11(b). Let $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ for $B, C, D, E \in M_n$. We have $AW_J = \begin{bmatrix} \lambda B & B+\lambda C \\ \lambda D & D+\lambda E \end{bmatrix}$ while $W_J A = \begin{bmatrix} \lambda B+D & \lambda C+E \\ \lambda D & \lambda E \end{bmatrix}$. These agree if and only if $\{\lambda B = \lambda B + D, B + \lambda C = \lambda C + E, D + \lambda E = \lambda E\}$, which hold if and only if $\{D = 0, B = E\}$, as desired.

7. Let $A \in M_n$. Prove that A commutes with $J_n(0)$ if and only if A is an upper triangular Toeplitz matrix.

See Ex. 3.2.2, 3.2.3, 3.2.4. Since $J_n(0)$ is nonderogatory, we apply Thm 3.2.4.2. A commutes with $J_n(0)$ if and only if there is some polynomial $p(t)$ of degree at most $n-1$ such that $A = p(J_n(0))$. This holds if and only if there are scalars $b_0, b_1, \ldots, b_{n-1}$ such that $A = b_0 I + b_1 J_n(0)^1 + b_2 J_n(0)^2 + \cdots + b_{n-1} J_n(0)^{n-1}$, which in turn holds if and only if $A = \begin{pmatrix} b_0 & b_1 & \cdots & b_{n-1} \\ 0 & b_0 & \cdots & b_{n-2} \\ & & \ddots & \\ 0 & 0 & \cdots & b_0 \end{pmatrix}$, an upper triangular Toeplitz matrix.